

# GENERALIZED MONOTONE TRIANGLES: AN EXTENDED COMBINATORIAL RECIPROCITY THEOREM

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**ABSTRACT.** In a recent work, the combinatorial interpretation of the polynomial  $\alpha(n; k_1, k_2, \dots, k_n)$  counting the number of Monotone Triangles with bottom row  $k_1 < k_2 < \dots < k_n$  was extended to weakly decreasing sequences  $k_1 \geq k_2 \geq \dots \geq k_n$ . In this case the evaluation of the polynomial is equal to a signed enumeration of objects called Decreasing Monotone Triangles. In this paper we define Generalized Monotone Triangles – a joint generalization of both ordinary Monotone Triangles and Decreasing Monotone Triangles. As main result of the paper we prove that the evaluation of  $\alpha(n; k_1, k_2, \dots, k_n)$  at arbitrary  $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$  is a signed enumeration of Generalized Monotone Triangles with bottom row  $(k_1, k_2, \dots, k_n)$ . Computational experiments indicate that certain evaluations of the polynomial at integral sequences yield well-known round numbers related to Alternating Sign Matrices. The main result provides a combinatorial interpretation of the conjectured identities and could turn out useful in giving a bijective proof.

## 1. INTRODUCTION

A *Monotone Triangle* of size  $n$  is a triangular array of integers  $(a_{i,j})_{1 \leq j \leq i \leq n}$

$$\begin{array}{ccccccc} & & & & a_{1,1} & & \\ & & & & a_{2,1} & & a_{2,2} \\ & & \ddots & & & & \ddots \\ & & & \ddots & & & \\ a_{n,1} & & \dots & & \dots & & a_{n,n} \end{array}$$

with strict increase along rows and weak increase along North-East- and South-East-diagonals, i.e.  $a_{i,j} < a_{i,j+1}$ ,  $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$ . An example of a Monotone Triangle of size 5 is given in Fig.1.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & & 4 & & 5 & & \\ & & & 3 & 5 & & 7 & & \\ & & 2 & 5 & 6 & & 8 & & \\ 2 & & 4 & 5 & 8 & & 9 & & \end{array}$$

FIGURE 1. One of the 16939 Monotone Triangles with bottom row  $(2, 4, 5, 8, 9)$ .

*Key words and phrases.* Combinatorial Reciprocity, Monotone Triangle, Generalized Monotone Triangle, Alternating Sign Matrix.

Supported by the Austrian Science Foundation FWF, START grant Y463.

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For each  $n \geq 1$ , there exists a unique polynomial  $\alpha(n; k_1, k_2, \dots, k_n)$  of degree  $n - 1$  in each of the  $n$  variables such that the evaluation of this polynomial at strictly increasing sequences  $k_1 < k_2 < \dots < k_n$  is equal to the number of Monotone Triangles with prescribed bottom row  $(k_1, k_2, \dots, k_n)$  – for example  $\alpha(5; 2, 4, 5, 8, 9) = 16939$ . This result was derived in [Fis06], where the polynomials are given explicitly in terms of an operator formula.

In [FR11] we studied the evaluation of  $\alpha(n; k_1, \dots, k_n)$  at weakly decreasing sequences  $k_1 \geq k_2 \geq \dots \geq k_n$ . It turned out that the evaluation can be interpreted as signed enumeration of the following combinatorial objects:

A *Decreasing Monotone Triangle* (DMT) of size  $n$  is a triangular array of integers  $(a_{i,j})_{1 \leq j \leq i \leq n}$  having the following properties:

- The entries along North-East- and South-East-diagonals are weakly decreasing.
- Each integer appears at most twice in a row.
- Two consecutive rows do not contain the same integer exactly once.

One of the motivations for considering evaluations of  $\alpha(n; k_1, \dots, k_n)$  at non-increasing  $(k_1, \dots, k_n) \in \mathbb{Z}^n$  stems from the connection to Alternating Sign Matrices. An *Alternating Sign Matrix* (ASM) of size  $n$  is a  $n \times n$ -matrix with entries in  $\{0, 1, -1\}$  such that in each row and column the non-zero entries alternate in sign and sum up to 1. It is well-known that the set of ASMs is in bijection with the set of Monotone Triangles with bottom row  $(1, 2, \dots, n)$ . Counting the number of ASMs of size  $n$  had been an open problem for more than a decade until the first two independent proofs were given by D. Zeilberger ([Zei96]) and G. Kuperberg ([Kup96]) in 1996 (see [Bre99] for more details). The Refined ASM Theorem – i.e. the refined enumeration with respect to the unique 1 in the first row – was reproven by I. Fischer in 2007 ([Fis07]). The identity

$$\alpha(n; k_1, \dots, k_n) = (-1)^{n-1} \alpha(n; k_2, \dots, k_n, k_1 - n) \quad (1.1)$$

plays one of the key roles in this algebraic proof. A bijective proof of (1.1) could give more combinatorial insight to the theorem. However, note that if  $k_1 < k_2 < \dots < k_n$ , then  $k_n > k_1 - n$ , i.e. (1.1) can per se only be understood as identity satisfied by the polynomial.

The objective of this paper is to give an interpretation to the evaluation of  $\alpha(n; k_1, \dots, k_n)$  at arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ . For this, we define triangular arrays of integers which locally combine the restrictions of ordinary Monotone Triangles and Decreasing Monotone Triangles:

A *Generalized Monotone Triangle* (GMT) is a triangular array  $(a_{i,j})_{1 \leq j \leq i \leq n}$  of integers satisfying the following conditions:

- (1) Each entry is weakly bounded by its SW- and SE-neighbour, i.e.

$$\min\{a_{i+1,j}, a_{i+1,j+1}\} \leq a_{i,j} \leq \max\{a_{i+1,j}, a_{i+1,j+1}\}.$$

- (2) If three consecutive entries in a row are weakly increasing, then their two interlaced neighbours in the row above are strictly increasing, i.e.

$$a_{i+1,j} \leq a_{i+1,j+1} \leq a_{i+1,j+2} \rightarrow a_{i,j} < a_{i,j+1}.$$

- (3) If two consecutive entries in a row are strictly decreasing and their interlaced neighbour in the row above is equal to its SW-/SE-neighbour, then the interlaced neighbour has a

left/right neighbour and is equal to it, i.e.

$$a_{i,j} = a_{i+1,j} > a_{i+1,j+1} \rightarrow a_{i,j-1} = a_{i,j},$$

$$a_{i+1,j} > a_{i+1,j+1} = a_{i,j} \rightarrow a_{i,j+1} = a_{i,j}.$$

By way of illustration, let us find all GMTs with bottom row  $(4, 2, 1, 3)$ : First, construct all possible penultimate rows  $(l_1, l_2, l_3)$ . Condition (1) implies that  $l_1 \in \{2, 3, 4\}$ , Condition (3) further restricts it to  $l_1 \in \{2, 3\}$ . If on the one hand  $l_1 = 2$ , then Condition (3) forces  $l_2 = 2$ . The right-most entry  $l_3$  is bounded by 1 and 3, but actually  $l_1 = l_2 = l_3 = 2$  would violate Condition (2), so  $l_3 \in \{1, 3\}$ . If on the other hand  $l_1 = 3$ , then Condition (3) implies that  $l_2 = l_3 = 1$ . Continuing in the same way with all penultimate rows yields the four GMTs depicted in Figure 2.

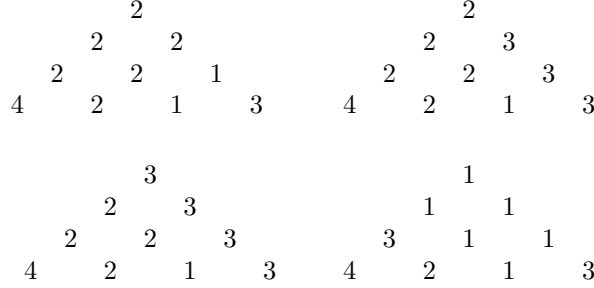


FIGURE 2. The four GMTs with bottom row  $(4, 2, 1, 3)$ .

For  $k_1 < k_2 < \dots < k_n$ , the set of GMTs with bottom row  $(k_1, \dots, k_n)$  is equal to the set of Monotone Triangles with this bottom row: Every GMT with strictly increasing bottom row is by conditions (1) and (2) a Monotone Triangle. Conversely, the weak increase along NE- and SE-diagonals of Monotone Triangles implies condition (1) of GMTs, the strict increase condition (2), and the premise of (3) can not hold.

For  $k_1 \geq k_2 \geq \dots \geq k_n$ , the set of GMTs with bottom row  $(k_1, \dots, k_n)$  is equal to the set of Decreasing Monotone Triangles with this bottom row: The NE- and SE-diagonals of every GMT with weakly decreasing bottom row are by condition (1) weakly decreasing. This also implies a weak decrease along rows, and thus three consecutive equal entries in a row would contradict condition (2). Furthermore, two consecutive rows containing an integer exactly once would contradict condition (3). Conversely, the weak decrease of DMTs along NE- and SE-diagonals implies condition (1) and weak decrease along rows. Thus, the premise of (2) can only hold if three consecutive entries coincide, which is not admissible in DMTs. Finally, condition (3) follows from the weak decrease along rows together with the condition that two consecutive rows do not contain the same entry exactly once.

Therefore, Generalized Monotone Triangles are indeed a joint generalization of ordinary Monotone Triangles and Decreasing Monotone Triangles. The main result of the paper is that the evaluation  $\alpha(n; k_1, \dots, k_n)$  is a signed enumeration of the GMTs with bottom row  $(k_1, k_2, \dots, k_n)$ . The sign of a GMT is determined by the following two statistics:

- (1) An entry  $a_{i,j}$  is called *newcomer* if  $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$ .

- (2) A pair  $(x, x)$  of two consecutive equal entries in a row is called *sign-changing*, if their interlaced neighbour in the row below is also equal to  $x$ .

Let  $\mathcal{G}_n(k_1, k_2, \dots, k_n)$  denote the set of GMTs with bottom row  $(k_1, k_2, \dots, k_n)$ .

**Theorem 1.** *Let  $n \geq 1$  and  $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ . Then*

$$\alpha(n; k_1, \dots, k_n) = \sum_{A \in \mathcal{G}_n(k_1, \dots, k_n)} (-1)^{\text{sc}(A)},$$

where  $\text{sc}(A)$  is the total number of newcomers and sign-changing pairs in  $A$ .

Applying Theorem 1 to our example in Figure 2 yields  $\alpha(4; 4, 2, 1, 3) = -2$ .

Theorem 1 is known to be true for strictly increasing sequences  $k_1 < k_2 < \dots < k_n$ , as in this case the set  $\mathcal{G}_n(k_1, \dots, k_n)$  is equal to the set of Monotone Triangles with bottom row  $(k_1, k_2, \dots, k_n)$  and  $\text{sc}(A) = 0$  for every Monotone Triangle.

Lemma 3 of [FR11] implies the correctness of Theorem 1 for weakly decreasing bottom rows: In this case  $\mathcal{G}_n(k_1, \dots, k_n)$  is equal to the set of DMTs with bottom row  $(k_1, \dots, k_n)$  and the  $\text{sc}$ -functions coincide. K. Jochemko and R. Sanyal recently gave a proof of the theorem in this case from a geometric point of view ([JS12]).

In Section 2 we give a straight-forward proof of Theorem 1 using a recursion satisfied by  $\alpha(n; k_1, \dots, k_n)$ . In Section 3 a connection with a known generalization ([Fis11]) is established, which enables us to give a shorter, more subtle proof of Theorem 1. Apart from being a joint generalization of Monotone Triangles and DMTs, this generalization is more reduced in the sense that fewer cancellations occur in the signed enumerations than in previously known generalizations. In Section 4 we apply the theorem to give a combinatorial proof of an identity satisfied by  $\alpha(n; k_1, \dots, k_n)$  and provide a collection of open problems.

## 2. PROOF OF THEOREM 1

The number of Monotone Triangles with bottom row  $(k_1, \dots, k_n)$  can be counted recursively by determining all admissible penultimate rows  $(l_1, \dots, l_{n-1})$  and summing over the number of Monotone Triangles with these bottom rows. The polynomial  $\alpha(n; k_1, \dots, k_n)$  hence satisfies

$$\alpha(n; k_1, \dots, k_n) = \sum_{\substack{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1}, \\ k_1 \leq l_1 \leq k_2 \leq l_2 \leq \dots \leq k_{n-1} \leq l_{n-1} \leq k_n, \\ l_i < l_{i+1}}} \alpha(n-1; l_1, \dots, l_{n-1}) \quad (2.1)$$

for all  $k_1 < k_2 < \dots < k_n$ ,  $k_i \in \mathbb{Z}$ . In fact ([Fis06]), one can define a summation operator  $\sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)}$  for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$  such that

$$\alpha(n; k_1, \dots, k_n) = \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} \alpha(n-1; l_1, \dots, l_{n-1}) \quad (2.2)$$

holds. The summation operator is defined recursively for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ :

$$\begin{aligned} \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} A(l_1, \dots, l_{n-1}) &:= \sum_{(l_1, \dots, l_{n-2})}^{(k_1, \dots, k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1, \dots, l_{n-2}, l_{n-1}) \\ &+ \sum_{(l_1, \dots, l_{n-2})}^{(k_1, \dots, k_{n-2}, k_{n-1}-1)} A(l_1, \dots, l_{n-2}, k_{n-1}), \quad n \geq 2, \end{aligned} \quad (2.3)$$

with  $\sum_{() }^{(k_1)} := \text{id}$  and the extended definition of simple sums

$$\sum_{i=a}^b f(i) := \begin{cases} 0, & b = a - 1, \\ - \sum_{i=b+1}^{a-1} f(i), & b + 1 \leq a - 1. \end{cases} \quad (2.4)$$

Using induction and (2.1), it is clear that (2.2) holds for increasing sequences  $k_1 < k_2 < \dots < k_n$ .

To prove it for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ , let us first note that applying  $\sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)}$  to a polynomial in  $(l_1, \dots, l_{n-1})$  yields a polynomial in  $(k_1, \dots, k_n)$ : In the base case  $n = 2$ , write the polynomial  $p(l_1)$  in terms of the binomial basis  $p(l_1) = \sum_{i=0}^{n-1} c_i \binom{l_1}{i}$ . The polynomial  $q(x) := \sum_{i=1}^n c_{i-1} \binom{x}{i}$  then satisfies  $q(x+1) - q(x) = p(x)$ . For integers  $a \leq b$ , it follows that  $\sum_{l_1=a}^b p(l_1) = q(b+1) - q(a)$ , but this is by definition (2.4) true for arbitrary  $a, b \in \mathbb{Z}$ . The inductive step is immediate using (2.3). Thus, we know that the right-hand side of (2.2) is a polynomial in  $(k_1, \dots, k_n)$  coinciding with the polynomial on the left-hand side whenever  $k_1 < k_2 < \dots < k_n$ . Since a polynomial in  $n$  variables is uniquely determined by these values, it follows that (2.2) indeed holds. The same is true for the alternative recursive description

$$\begin{aligned} \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} A(l_1, \dots, l_{n-1}) &= \sum_{(l_1, \dots, l_{n-2})}^{(k_1, \dots, k_{n-1})} \sum_{l_{n-1}=k_{n-1}}^{k_n} A(l_1, \dots, l_{n-2}, l_{n-1}) \\ &- \sum_{(l_1, \dots, l_{n-3})}^{(k_1, \dots, k_{n-2})} A(l_1, \dots, l_{n-3}, k_{n-1}, k_{n-1}), \quad n \geq 3. \end{aligned} \quad (2.5)$$

The following Lemma establishes a connection between the summation operator and GMTs, which then gives us the means to prove Theorem 1 inductively.

**Lemma 1.** *Let  $\mathcal{P}(k_1, \dots, k_n)$  denote the set of  $(n-1)$ -st rows of elements in  $\mathcal{G}_n(k_1, k_2, \dots, k_n)$ . Then every function  $A(l_1, \dots, l_{n-1})$  satisfies*

$$\sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} A(l_1, \dots, l_{n-1}) = \sum_{(l_1, \dots, l_{n-1}) \in \mathcal{P}(k_1, \dots, k_n)} (-1)^{\text{sc}(\mathbf{k}; \mathbf{l})} A(l_1, \dots, l_{n-1}), \quad n \geq 2,$$

where  $\text{sc}(\mathbf{k}; \mathbf{l}) := \text{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1})$  is the total number of newcomers and sign-changing pairs in  $(l_1, \dots, l_{n-1})$ .

Before proving the Lemma, let us first give a remark, which is solely based on the definition of GMTs. In general, the set of admissible values for an entry  $l_i$  depends on its neighbours  $l_{i-1}$  and  $l_{i+1}$  as well as the four adjacent entries  $k_{i-1}$ ,  $k_i$ ,  $k_{i+1}$  and  $k_{i+2}$  in the row below – ordered

$$\begin{array}{ccccccc} & & l_{i-1} & & l_i & & l_{i+1} \\ & & & & & & \\ k_{i-1} & & & k_i & & k_{i+1} & k_{i+2} \end{array}$$

– in the following way: If  $k_{i-1} > l_{i-1} = k_i$ , then the only admissible value is  $l_i = k_i$ . Symmetrically, if  $k_{i+1} = l_{i+1} > k_{i+2}$ , then  $l_i = k_{i+1}$ . Otherwise,  $l_i$  can take any value strictly between  $k_i$  and  $k_{i+1}$ . To determine, whether  $l_i = k_i$  is allowed, check whether  $k_i > k_{i+1}$ , or  $k_{i-1} > k_i \leq k_{i+1}$  or  $k_{i-1} \leq k_i \leq k_{i+1}$ . If  $k_i > k_{i+1}$ , then  $l_i = k_i$  is admissible, if and only if  $l_{i-1} = k_i$ . If  $k_{i-1} > k_i \leq k_{i+1}$ , then  $l_i = k_i$  is admissible. If  $k_{i-1} \leq k_i \leq k_{i+1}$ , then  $l_i = k_i$  is admissible, if and only if  $l_{i-1} < k_i$ . Determining whether  $l_i = k_{i+1}$  is admissible works symmetrically.

*Proof.* If  $n = 2$ , then the result is immediate using (2.4). For  $n = 3$ , let us check the case  $k_1 \geq k_2$ ,  $k_2 < k_3$  (the other cases can be shown in the same way):

$$\begin{aligned} \sum_{(l_1, l_2)}^{(k_1, k_2, k_3)} A(l_1, l_2) &\stackrel{(2.5)}{=} \sum_{(l_1)}^{(k_1, k_2)} \sum_{l_2=k_2}^{k_3} A(l_1, l_2) - \sum_{()}^{(k_1)} A(k_2, k_2) \\ &\stackrel{(2.3)}{=} \sum_{()}^{(k_1)} \sum_{l_1=k_1+1}^{k_2} \sum_{l_2=k_2}^{k_3} A(l_1, l_2) + \sum_{()}^{(k_1-1)} \sum_{l_2=k_2}^{k_3} A(k_1, l_2) - A(k_2, k_2) \\ &= \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} A(l_1, l_2) - A(k_2, k_2). \end{aligned}$$

If  $k_1 = k_2$ , then  $\mathcal{P}(k_1, k_2, k_3) = \{(k_1, l_2) : k_2 < l_2 \leq k_3\}$  with no newcomers or sign-changing pairs. If  $k_1 > k_2$ , then  $\mathcal{P}(k_1, k_2, k_3) = \{(l_1, l_2) : k_1 > l_1 > k_2, k_2 \leq l_2 \leq k_3\} \cup \{(k_2, k_2)\}$ . The entry  $l_1$  is either a newcomer or contained in a sign-changing pair. The claimed equation follows from (2.4).

For  $n \geq 4$ , we have to distinguish between the cases  $k_{n-1} \leq k_n$  (Case 1) and  $k_{n-1} > k_n$  (Case 2). The remark preceding the proof also suggests a different behaviour depending on whether  $l_{n-1}$  – the rightmost entry of the penultimate row – is equal to  $k_{n-1}$  or not. Indeed, this yields the sub-cases 1.1, 1.2 and 2.1, 2.2 respectively.

**Case 1 ( $k_{n-1} \leq k_n$ ) :**

Recursion (2.3) of the summation operator and the induction hypothesis yield

$$\begin{aligned}
& \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} A(l_1, \dots, l_{n-1}) \\
&= \sum_{(l_1, \dots, l_{n-2})}^{(k_1, \dots, k_{n-1})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1, \dots, l_{n-1}) + \sum_{(l_1, \dots, l_{n-2})}^{(k_1, \dots, k_{n-2}, k_{n-1}-1)} A(l_1, \dots, l_{n-2}, k_{n-1}) \\
&= \sum_{(l_1, \dots, l_{n-2}) \in \mathcal{P}(k_1, \dots, k_{n-1})} (-1)^{\text{sc}(k_1, \dots, k_{n-1}; l_1, \dots, l_{n-2})} \sum_{l_{n-1}=k_{n-1}+1}^{k_n} A(l_1, \dots, l_{n-1}) \\
&+ \sum_{(l_1, \dots, l_{n-2}) \in \mathcal{P}(k_1, \dots, k_{n-2}, k_{n-1}-1)} (-1)^{\text{sc}(k_1, \dots, k_{n-2}, k_{n-1}-1; l_1, \dots, l_{n-2})} A(l_1, \dots, l_{n-2}, k_{n-1}).
\end{aligned}$$

To see that this is further equal to

$$\sum_{(l_1, \dots, l_{n-1}) \in \mathcal{P}(k_1, \dots, k_n)} (-1)^{\text{sc}(\mathbf{k}; \mathbf{l})} A(l_1, \dots, l_{n-1}),$$

let us show that for  $k_{n-1} \leq k_n$

$$\begin{aligned}
& \mathcal{P}(k_1, \dots, k_n) \\
&= \mathcal{P}(k_1, \dots, k_{n-1}) \times \{l_{n-1} \mid k_{n-1} < l_{n-1} \leq k_n\} \cup \mathcal{P}(k_1, \dots, k_{n-2}, k_{n-1}-1) \times \{k_{n-1}\} \quad (2.6)
\end{aligned}$$

holds, and that each fixed row causes the same total number of sign-changes on the left-hand side as on the right-hand side.

Since the first  $n-2$  entries of the bottom row are identical on both sides of (2.6), it suffices – by the remark preceding the proof – to show that the restrictions imposed on  $l_{n-1}$ ,  $l_{n-2}$  and  $l_{n-3}$  are the same on both sides. For this, consider the entry  $l_{n-1}$  and distinguish between  $k_{n-1} < l_{n-1} \leq k_n$  and  $l_{n-1} = k_{n-1}$ :

**Case 1.1** ( $k_{n-1} < l_{n-1} \leq k_n$ ) :

If  $k_{n-2} > k_{n-1}$ , then  $k_{n-2} \geq l_{n-2} > k_{n-1}$  on both sides:

Left-hand side of (2.6)				
	$l_{n-2}$		$l_{n-1}$	
	$\nearrow$	$\searrow$	$\swarrow$	$\nwarrow$
$k_{n-2}$	$>$	$k_{n-1}$	$\leq$	$k_n$
Right-hand side of (2.6)				
	$l_{n-2}$		$l_{n-1}$	
	$\nearrow$	$\searrow$		
$k_{n-2}$	$>$	$k_{n-1}$		

The restrictions for  $l_{n-3}$  are the same on both sides. The entry  $l_{n-1}$  does not contribute a sign-change, and the entry  $l_{n-2}$  is involved in a sign-change on both sides.

If  $k_{n-2} \leq k_{n-1}$ , then  $k_{n-2} \leq l_{n-2} \leq k_{n-1}$  on both sides:

Left-hand side of (2.6)				
	$l_{n-2}$		$l_{n-1}$	
$k_{n-2}$	$\leq$	$k_{n-1}$	$\leq$	$k_n$
Right-hand side of (2.6)				
	$l_{n-2}$		$l_{n-1}$	
$k_{n-2}$	$\leq$	$k_{n-1}$		

The restrictions for  $l_{n-3}$  are the same on both sides. The entry  $l_{n-1}$  does not contribute a sign-change, and the entry  $l_{n-2}$  is involved in a sign-change on the left-hand side, if and only if it is on the right-hand side.

It follows that  $\text{sc}(k_1, \dots, k_{n-1}; l_1, \dots, l_{n-2}) = \text{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1})$ .

**Case 1.2 ( $l_{n-1} = k_{n-1}$ ) :**

If  $k_{n-2} = k_{n-1}$ , then there is no row on the left-hand side with  $l_{n-1} = k_{n-1}$ , and on the right-hand side this would imply  $l_{n-3} = l_{n-2} = l_{n-1} = k_{n-1}$ :

Left-hand side of (2.6)				
	$\not\leq$		$k_{n-1}$	
$k_{n-2}$	$=$	$k_{n-1}$	$\leq$	$k_n$
Right-hand side of (2.6)				
$k_{n-3}$	$=$	$k_{n-1}$	$>$	$k_{n-1} - 1$

But since a GMT can not contain three consecutive equal entries, such rows are not contained on the right-hand side.

If  $k_{n-2} \leq k_{n-1} - 1$ , then  $k_{n-2} \leq l_{n-2} < k_{n-1}$  on both sides:

Left-hand side of (2.6)				
	$l_{n-2}$		$k_{n-1}$	
$k_{n-2}$	$<$	$k_{n-1}$	$\leq$	$k_n$
Right-hand side of (2.6)				
	$l_{n-2}$		$k_{n-1}$	
$k_{n-2}$	$\leq$	$k_{n-1} - 1$		



The restrictions for  $l_{n-3}$  are the same on both sides. The entry  $l_{n-1}$  does not contribute a sign-change, and the entry  $l_{n-2}$  is involved in a sign-change on the left-hand side, if and only if it is on the right-hand side.

If  $k_{n-2} > k_{n-1}$ , then  $k_{n-2} \geq l_{n-2} \geq k_{n-1}$  on both sides:

Left-hand side of (2.6)				
	$l_{n-2}$		$k_{n-1}$	
$k_{n-2}$	$\nearrow$	$\searrow$	$\parallel$	$\nwarrow$
	$>$		$\leq$	
		$k_{n-1}$		$k_n$
Right-hand side of (2.6)				
	$l_{n-2}$		$k_{n-1}$	
$k_{n-2}$	$\nearrow$	$\searrow$		
	$>$		$k_{n-1} - 1$	

The restrictions for  $l_{n-3}$  are the same on both sides. The entry  $l_{n-2}$  is involved in a sign-change on both sides (note the special case  $l_{n-2} = k_{n-1}$ , where  $l_{n-2}$  is part of a sign-changing pair on the left-hand side and a newcomer on the right-hand side).

It follows that  $\text{sc}(k_1, \dots, k_{n-2}, k_{n-1} - 1; l_1, \dots, l_{n-2}) = \text{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1})$ .

**Case 2 ( $k_{n-1} > k_n$ ) :**

Recursion (2.5) of the summation operator and the induction hypothesis yield

$$\begin{aligned}
& \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} A(l_1, \dots, l_{n-1}) \\
&= - \sum_{(l_1, \dots, l_{n-2})}^{(k_1, \dots, k_{n-1})} \sum_{l_{n-1}=k_n+1}^{k_{n-1}-1} A(l_1, \dots, l_{n-1}) - \sum_{(l_1, \dots, l_{n-3})}^{(k_1, \dots, k_{n-2})} A(l_1, \dots, l_{n-3}, k_{n-1}, k_{n-1}) \\
&= \sum_{(l_1, \dots, l_{n-2}) \in \mathcal{P}(k_1, \dots, k_{n-1})} (-1)^{\text{sc}(k_1, \dots, k_{n-1}; l_1, \dots, l_{n-2})+1} \sum_{l_{n-1}=k_n+1}^{k_{n-1}-1} A(l_1, \dots, l_{n-1}) \\
&\quad + \sum_{(l_1, \dots, l_{n-3}) \in \mathcal{P}(k_1, \dots, k_{n-2})} (-1)^{\text{sc}(k_1, \dots, k_{n-2}; l_1, \dots, l_{n-3})+1} A(l_1, \dots, l_{n-3}, k_{n-1}, k_{n-1}).
\end{aligned}$$

Similarly, let us show that for  $k_{n-1} > k_n$

$$\begin{aligned}
& \mathcal{P}(k_1, \dots, k_n) \\
&= \mathcal{P}(k_1, \dots, k_{n-1}) \times \{l_{n-1} \mid k_{n-1} > l_{n-1} > k_n\} \cup \mathcal{P}(k_1, \dots, k_{n-2}) \times \{(k_{n-1}, k_{n-1})\} \quad (2.7)
\end{aligned}$$

holds. Again it suffices to show that  $l_{n-1}$ ,  $l_{n-2}$  and  $l_{n-3}$  have to satisfy the same restrictions on both sides, and that corresponding rows contain the same number of sign-changes. Since  $k_{n-1} > k_n$ , it follows that  $k_{n-1} \geq l_{n-1} > k_n$  on both sides. Let us distinguish between the cases  $k_{n-1} > l_{n-1} > k_n$  and  $l_{n-1} = k_{n-1}$ :

**Case 2.1 ( $k_{n-1} > l_{n-1} > k_n$ ) :**

If  $k_{n-2} > k_{n-1}$ , then  $k_{n-2} \geq l_{n-2} > k_{n-1}$  on both sides:

Left-hand side of (2.7)				
	$l_{n-2}$		$l_{n-1}$	
$\nearrow$		$\searrow$	$\nearrow$	$\searrow$
$k_{n-2}$	$>$	$k_{n-1}$	$>$	$k_n$
Right-hand side of (2.7)				
	$l_{n-2}$		$l_{n-1}$	
$\nearrow$		$\searrow$		
$k_{n-2}$	$>$	$k_{n-1}$		

The restrictions for  $l_{n-3}$  are the same on both sides. The entries  $l_{n-1}$  and  $l_{n-2}$  both contribute a sign-change.

If  $k_{n-2} \leq k_{n-1}$ , then  $k_{n-2} \leq l_{n-2} \leq k_{n-1}$  on both sides:

Left-hand side of (2.7)				
	$l_{n-2}$		$l_{n-1}$	
$\swarrow$		$\searrow$	$\nearrow$	$\searrow$
$k_{n-2}$	$\leq$	$k_{n-1}$	$>$	$k_n$
Right-hand side of (2.7)				
	$l_{n-2}$		$l_{n-1}$	
$\swarrow$		$\searrow$		
$k_{n-2}$	$\leq$	$k_{n-1}$		

The restrictions for  $l_{n-3}$  are the same on both sides. The entry  $l_{n-1}$  contributes a sign-change, and the entry  $l_{n-2}$  is involved in a sign-change on the left-hand side, if and only if it is on the right-hand side.

It follows that  $\text{sc}(k_1, \dots, k_{n-1}; l_1, \dots, l_{n-2}) + 1 = \text{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1})$ .

**Case 2.2 ( $l_{n-1} = k_{n-1}$ ) :**

Since  $k_{n-1} > k_n$  and  $l_{n-1} = k_{n-1}$ , we have  $l_{n-2} = k_{n-1}$  on both sides, whereby  $(l_{n-2}, l_{n-1})$  is a sign-changing pair. It remains to be shown that  $l_{n-3}$  has the same restrictions on both sides.

If  $k_{n-3} \leq k_{n-2}$ , then  $k_{n-3} \leq l_{n-3} \leq k_{n-2}$  on the right-hand side:

Left-hand side of (2.7)				
	$l_{n-3}$	$k_{n-1}$	$k_{n-1}$	
$\swarrow$		$\parallel$	$\parallel$	$\searrow$
$k_{n-3}$	$\leq$	$k_{n-2}$	$k_{n-1}$	$>$
				$k_n$
Right-hand side of (2.7)				
	$l_{n-3}$	$k_{n-1}$	$k_{n-1}$	
$\swarrow$		$\searrow$		
$k_{n-3}$	$\leq$	$k_{n-2}$		

On the left-hand side we also have  $k_{n-3} \leq l_{n-3} \leq k_{n-2}$ , unless  $k_{n-2} = k_{n-1}$ . In this case  $k_{n-3} \leq l_{n-3} < k_{n-2}$ , but for  $l_{n-3} = k_{n-2} = k_{n-1} = l_{n-2} = l_{n-1}$  there are three consecutive equal entries anyway. The entry  $l_{n-3}$  is involved in a sign-change on the left-hand side, if and only if it is on the right-hand side, and  $(l_{n-2}, l_{n-1})$  is a sign-changing pair.

If  $k_{n-3} > k_{n-2}$  (and  $n > 4$ ), then  $k_{n-3} \geq l_{n-3} > k_{n-2}$  on the right-hand side:

Left-hand side of (2.7)					
	$l_{n-3}$		$k_{n-1}$		$k_{n-1}$
	$\nearrow$		$\parallel$		$\searrow$
$k_{n-3}$	$>$	$k_{n-2}$	$\parallel$	$k_{n-1}$	$>$
					$k_n$
Right-hand side of (2.7)					
	$l_{n-3}$		$k_{n-1}$		$k_{n-1}$
	$\nearrow$		$\searrow$		
$k_{n-3}$	$>$	$k_{n-2}$			

Again,  $l_{n-3}$  has the same restrictions on the left-hand side, unless  $k_{n-2} = k_{n-1}$ . In this case  $k_{n-3} \geq l_{n-3} \geq k_{n-2}$ , whereby  $l_{n-3} = k_{n-2}$  implies that  $l_{n-3} = k_{n-2} = k_{n-1} = l_{n-2} = l_{n-1}$ . If  $n = 4$ , then the same holds with the difference that  $k_1 > l_1$  instead of  $k_1 \geq l_1$  on both sides. The entry  $l_{n-3}$  is involved in a sign-change on both sides and  $(l_{n-2}, l_{n-1})$  is a sign-changing pair.

It follows that  $\text{sc}(k_1, \dots, k_{n-2}; l_1, \dots, l_{n-3}) + 1 = \text{sc}(k_1, \dots, k_n; l_1, \dots, l_{n-1})$ .

□

*Proof (Theorem 1).* The result is immediate for  $n = 1$ :

$$\alpha(1; k_1) = 1 = \sum_{A \in \mathcal{G}_1(k_1)} (-1)^{\text{sc}(A)}.$$

For  $n \geq 2$  apply (2.2), Lemma 1 and the induction hypothesis:

$$\begin{aligned}
\alpha(n; k_1, \dots, k_n) &= \sum_{(l_1, \dots, l_{n-1})}^{(k_1, \dots, k_n)} \alpha(n-1; l_1, \dots, l_{n-1}) \\
&= \sum_{(l_1, \dots, l_{n-1}) \in \mathcal{P}(k_1, \dots, k_n)} (-1)^{\text{sc}(\mathbf{k}; \mathbf{l})} \alpha(n-1; l_1, \dots, l_{n-1}) \\
&= \sum_{(l_1, \dots, l_{n-1}) \in \mathcal{P}(k_1, \dots, k_n)} (-1)^{\text{sc}(\mathbf{k}; \mathbf{l})} \sum_{A \in \mathcal{G}_{n-1}(l_1, \dots, l_{n-1})} (-1)^{\text{sc}(A)} = \sum_{A \in \mathcal{G}_n(k_1, \dots, k_n)} (-1)^{\text{sc}(A)}.
\end{aligned}$$

□

### 3. CONNECTION WITH DIFFERENT GENERALIZATION & ALTERNATIVE PROOF

In [Fis11] four different combinatorial extensions of  $\alpha(n; k_1, \dots, k_n)$  to all  $(k_1, \dots, k_n) \in \mathbb{Z}^n$  are described. The idea behind all of them is to write the sum in (2.1) in terms of simple summations, i.e. summations as defined in (2.4). In the third extension this is based on the inclusion-exclusion principle: Let  $k_1 < k_2 < \dots < k_n$  and

$$\begin{aligned} M &:= \{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \leq l_j \leq k_{j+1} \wedge l_j < l_{j+1}\}, \\ A &:= \{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \leq l_j \leq k_{j+1}\}, \\ A_i &:= \{(l_1, \dots, l_{n-1}) \in \mathbb{Z}^{n-1} \mid \forall j : k_j \leq l_j \leq k_{j+1} \wedge l_{i-1} = k_i = l_i\}, \quad i = 2, \dots, n-1. \end{aligned}$$

The strict increase implies that  $A_i \cap A_{i+1} = \emptyset$ , and thus we have for any function  $f(\mathbf{l}) := f(l_1, \dots, l_{n-1})$  that

$$\begin{aligned} \sum_{\mathbf{l} \in M} f(\mathbf{l}) &= \sum_{\mathbf{l} \in A} f(\mathbf{l}) - \sum_{i=2}^{n-1} \sum_{\mathbf{l} \in A_i} f(\mathbf{l}) + \sum_{\substack{2 \leq i_1 < i_2 \leq n-1 \\ i_2 \neq i_1+1}} \sum_{\mathbf{l} \in A_{i_1} \cap A_{i_2}} f(\mathbf{l}) \\ &\quad - \sum_{\substack{2 \leq i_1 < i_2 < i_3 \leq n-1 \\ i_{j+1} \neq i_j+1}} \sum_{\mathbf{l} \in A_{i_1} \cap A_{i_2} \cap A_{i_3}} f(\mathbf{l}) \cdots, \end{aligned} \quad (3.1)$$

which can be written in terms of simple sums as

$$\sum_{p \geq 0} (-1)^p \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_p \leq n-1 \\ i_{j+1} \neq i_j+1}} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \cdots \sum_{l_{i_1}-1=k_{i_1}}^{k_{i_1}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p}-1=k_{i_p}}^{k_{i_p}} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1}=k_{n-1}}^{k_n} f(\mathbf{l}). \quad (3.2)$$

Using (2.4), we can interpret (3.2) for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ . Let us show that

$$\begin{aligned} \alpha(n; k_1, \dots, k_n) &= \sum_{p \geq 0} (-1)^p \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_p \leq n-1 \\ i_{j+1} \neq i_j+1}} \\ &\quad \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \cdots \sum_{l_{i_1}-1=k_{i_1}}^{k_{i_1}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p}-1=k_{i_p}}^{k_{i_p}} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1}=k_{n-1}}^{k_n} \alpha(n-1; l_1, \dots, l_{n-1}) \end{aligned} \quad (3.3)$$

holds for  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ . The correctness for  $k_1 < k_2 < \dots < k_n$  is ensured by (2.1), (3.1) and (3.2). To prove it for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ , it thus suffices to show that (3.2) applied to a polynomial in  $l_1, \dots, l_{n-1}$  yields a polynomial in  $k_1, \dots, k_n$ . But this follows from (2.4) in the exact same way as in the proof of (2.2).

As pointed out in [Fis11], we can give (3.3) a combinatorial meaning by interpreting  $\alpha(n; k_1, \dots, k_n)$  as signed enumeration of the following combinatorial objects: In a triangular array  $(a_{i,j})_{1 \leq j \leq i \leq n}$  of integers, let us call the entries  $a_{i-1,j-1}$  and  $a_{i-1,j}$  the parents of  $a_{i,j}$ . Among the entries  $(a_{i,j})_{1 \leq j \leq i \leq n}$ , there may be *special entries*. Special entries in the same row must not be adjacent (choosing these special entries corresponds to fixing the  $i_l$ 's in (3.3)). The requirements for the entries are

- (1) If  $a_{i,j}$  is special, then  $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$ .

- (2) If  $a_{i,j}$  is not the parent of a special entry and  $a_{i+1,j} \leq a_{i+1,j+1}$ , then  $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$ .
- (3) If  $a_{i,j}$  is not the parent of a special entry and  $a_{i+1,j} > a_{i+1,j+1}$ , then  $a_{i+1,j+1} > a_{i,j} > a_{i+1,j}$ .  
In this case  $a_{i,j}$  is called *inversion*.

Let us denote by  $\mathcal{T}_n(k_1, \dots, k_n)$  the set of these objects with bottom row  $(a_{n,1}, \dots, a_{n,n}) = (k_1, \dots, k_n)$ . For  $A \in \mathcal{T}_n(k_1, \dots, k_n)$  let  $s(A)$  be the total number of special entries and inversions. Using induction and (3.3), we thus have

$$\alpha(n; k_1, \dots, k_n) = \sum_{A \in \mathcal{T}_n(k_1, \dots, k_n)} (-1)^{s(A)}.$$

We can now eliminate those arrays  $(a_{i,j})_{1 \leq j \leq i \leq n}$  violating the condition

$$a_{i,j-1} \leq a_{i,j} \leq a_{i,j+1} \rightarrow a_{i-1,j-1} < a_{i-1,j} \quad (3.4)$$

by using the following sign-reversing involution: find the minimal index  $i$ , and under those the minimal index  $j$  such that  $a_{i,j-1} \leq a_{i,j} \leq a_{i,j+1}$  and  $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$ . If  $a_{i,j}$  is special, then turn it non-special, and vice-versa. Note that the minimality of  $i$  ensures that turning  $a_{i,j}$  special is admissible: Suppose a neighbour of  $a_{i,j}$  is special, then the row above contains three consecutive equal entries and thus an entry violating (3.4). It follows that

$$\alpha(n; k_1, \dots, k_n) = \sum_{\substack{A \in \mathcal{T}_n(k_1, \dots, k_n) \\ a_{i,j-1} \leq a_{i,j} \leq a_{i,j+1} \rightarrow a_{i-1,j-1} < a_{i-1,j}}} (-1)^{s(A)}.$$

Note that in this reduced set an entry  $a_{i,j}$  is special if and only if  $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$ . Hence, the additional information of which entries are special is not required anymore. Since special entries now correspond to sign-changing pairs and inversions to newcomers, the only remaining part for proving Theorem 1 is to show that

$$\mathcal{G}_n(k_1, \dots, k_n) = \{A \in \mathcal{T}_n(k_1, \dots, k_n) : a_{i,j-1} \leq a_{i,j} \leq a_{i,j+1} \rightarrow a_{i-1,j-1} < a_{i-1,j}\},$$

where an entry  $a_{i,j}$  is special if and only if  $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$ .

Let  $A \in \mathcal{G}_n(k_1, \dots, k_n)$ . Then two adjacent special entries in a row would imply three consecutive equal entries in a row, in contradiction to Condition (2) of GMTs. If  $a_{i,j}$  is special, then  $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$  by definition. If  $a_{i+1,j} \leq a_{i+1,j+1}$ , then  $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$  by Condition (1) of GMTs. If  $a_{i+1,j} > a_{i+1,j+1}$ , then  $a_{i+1,j} \geq a_{i,j} \geq a_{i+1,j+1}$  by Condition (1) of GMTs, and if  $a_{i+1,j}$  and  $a_{i+1,j+1}$  are neither special, Condition (3) of GMTs implies that  $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$ . We thus have  $A \in \mathcal{T}_n(k_1, \dots, k_n)$ , and the additional property is exactly Condition (2) of GMTs.

Let  $A \in \mathcal{T}_n(k_1, \dots, k_n)$  such that  $a_{i,j-1} \leq a_{i,j} \leq a_{i,j+1}$  implies  $a_{i-1,j-1} < a_{i-1,j}$ . Conditions (1) and (2) of GMTs are then trivially satisfied. If  $a_{i,j} = a_{i+1,j} > a_{i+1,j+1}$ , then by Condition (3) the entry  $a_{i,j}$  has to be parent of a special entry, and thus  $a_{i,j} = a_{i+1,j} = a_{i,j-1}$ . The second part of Condition (3) of GMTs is symmetric, and therefore  $A \in \mathcal{G}_n(k_1, \dots, k_n)$ .

This concludes the less straight-forward, yet much shorter proof of Theorem 1.

## 4. APPLICATIONS &amp; OPEN PROBLEMS

With this generalization at hand, we can try to give a combinatorial interpretation to identities satisfied by  $\alpha(n; k_1, \dots, k_n)$ . By way of illustration, take the identity

$$\begin{aligned} & \alpha(n; k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n) \\ &= \alpha(n; k_1, \dots, k_{i-1}, k_i, k_i, k_{i+2}, \dots, k_n) + \alpha(n; k_1, \dots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \dots, k_n). \end{aligned} \quad (4.1)$$

A combinatorial proof of this identity in the case that  $k_1 < k_2 < \dots < k_i$  and  $k_i + 1 < k_{i+2} < \dots < k_n$  was given in [Fis11]. Using Theorem 1, we can now give a combinatorial proof for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$  by showing that there exists a sign-preserving bijection

$$\begin{aligned} & \mathcal{G}_n(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n) \\ & \leftrightarrow \mathcal{G}_n(k_1, \dots, k_{i-1}, k_i, k_i, k_{i+2}, \dots, k_n) \dot{\cup} \mathcal{G}_n(k_1, \dots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \dots, k_n). \end{aligned}$$

If  $\mathcal{P}(k_1, \dots, k_n)$  denotes the set of penultimate rows of GMTs with bottom row  $(k_1, \dots, k_n)$ , it suffices to show that

$$\begin{aligned} & \mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n) \\ &= \mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i, k_{i+2}, \dots, k_n) \dot{\cup} \mathcal{P}(k_1, \dots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \dots, k_n), \end{aligned} \quad (4.2)$$

where each fixed row has the same total number of sign-changes on both sides.

Each  $(l_1, \dots, l_{n-1}) \in \mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n)$  satisfies  $l_i \in \{k_i, k_i + 1\}$ . Let us show that the set of penultimate rows with  $l_i = k_i$  is equal to  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i, k_{i+2}, \dots, k_n)$ . It is clear that  $l_i = k_i$  implies that the restrictions for  $(l_1, \dots, l_{i-1})$  are identical for both  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n)$  and  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i, k_{i+2}, \dots, k_n)$ . For the restrictions of  $(l_{i+1}, l_{i+2})$  distinguish between  $k_i + 1 \leq k_{i+2}$ ,  $k_i = k_{i+2}$  and  $k_i > k_{i+2}$ :

If  $k_i + 1 \leq k_{i+2}$ , then  $k_i + 1 \leq l_{i+1} \leq k_{i+2}$  on both sides and the restrictions for  $l_{i+2}$  are the same:

Left-hand side of (4.2)				
	$k_i$		$l_{i+1}$	
$k_i$	$<$	$k_i + 1$	$\leq$	$k_{i+2}$
Right-hand side of (4.2)				
	$k_i$		$l_{i+1}$	
$k_i$	$=$	$k_i$	$\leq$	$k_{i+2}$

If  $k_i = k_{i+2}$ , then  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i, k_{i+2}, \dots, k_n)$  is empty, and each element of  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n)$  with  $l_i = k_i$  would have to satisfy  $l_i = l_{i+1} = l_{i+2} = k_i$ :

Left-hand side of (4.2)				
	$k_i$		$k_i$	$k_i$
$\parallel$		$\nearrow$	$\searrow$	$\parallel$
$k_i$	$<$	$k_i + 1$	$>$	$k_i$
Right-hand side of (4.2)				
	$k_i$		$k_i$	
$\parallel$		$\parallel$	$\parallel$	
$k_i$	$=$	$k_i$	$=$	$k_i$

But, since a GMT can not contain three consecutive equal entries, there is also no element in  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_i, \dots, k_n)$  with  $l_i = k_i$ .

If  $k_i > k_{i+2}$ , then  $k_i \geq l_{i+1} \geq k_{i+2}$  on both sides and the restrictions for  $l_{i+2}$  are the same:

Left-hand side of (4.2)				
	$k_i$		$l_{i+1}$	
$\parallel$		$\nearrow$	$\searrow$	$\parallel$
$k_i$	$<$	$k_i + 1$	$>$	$k_{i+2}$
Right-hand side of (4.2)				
	$k_i$		$l_{i+1}$	
$\parallel$		$\parallel$	$\nearrow$	$\searrow$
$k_i$	$=$	$k_i$	$>$	$k_{i+2}$

The entry  $l_{i+1}$  is involved in a sign-change on both sides (note the special case  $l_{i+1} = k_i$ , where  $l_{i+1}$  is a newcomer on the left-hand side and in a sign-changing pair on the right-hand side).

The restrictions for  $(l_{i+3}, \dots, l_{n-1})$  are clearly the same for both sides. Symmetrically, one can also see that the set  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i, k_i + 1, k_{i+2}, \dots, k_n)$  restricted to  $l_i = k_{i+1}$  is the same as  $\mathcal{P}(k_1, \dots, k_{i-1}, k_i + 1, k_i + 1, k_{i+2}, \dots, k_n)$ , concluding the combinatorial proof of (4.1) for arbitrary  $(k_1, \dots, k_n) \in \mathbb{Z}^n$ .

A natural question could now be, whether similar identities hold if the difference between  $k_{i+1}$  and  $k_i$  is larger. For fixed integers  $k_1, \dots, k_{i-1}, k_{i+2}, \dots, k_n$ , let

$$t_n(k_i, k_{i+1}) := \alpha(n; k_1, \dots, k_{i-1}, k_i, k_{i+1}, k_{i+2}, \dots, k_n).$$

Similarly - with a bit more patience - one can also show the identity

$$\begin{aligned} t_n(k_i, k_i + 2) \\ = t_n(k_i, k_i) + t_n(k_i + 1, k_i + 1) + t_n(k_i + 2, k_i + 2) + t_n(k_i + 2, k_i + 1) + t_n(k_i + 1, k_i) \end{aligned} \quad (4.3)$$

combinatorially. Both (4.1) and (4.3) are special cases of the following identity: Let  $V_{x,y}$  be the operator defined as

$$V_{x,y}f(x, y) := f(x - 1, y) + f(x, y + 1) - f(x - 1, y + 1).$$

The function  $f_i(k_1, \dots, k_n) := V_{k_i, k_{i+1}}\alpha(n; k_1, \dots, k_n)$  then satisfies

$$f_i(k_1, \dots, k_n) = -f_i(k_1, \dots, k_{i-1}, k_{i+1} + 1, k_i - 1, k_{i+2}, \dots, k_n). \quad (4.4)$$

Setting  $k_{i+1} = k_i - 1$  in (4.4) immediately implies (4.1). Equation (4.3) is then the special case  $k_{i+1} = k_i - 2$  in (4.4). A similar shift-antisymmetry property for *Gelfand-Tsetlin Patterns* (Monotone Triangles without the condition of strict increase along rows) was shown bijectively in a recent work ([Fis11]). It would be interesting to give a bijective proof of (4.4) in the general case (an algebraic proof was given in [Fis06]).

In [FR11] we showed the surprising identity

$$A_n := \alpha(n; 1, 2, \dots, n) = \alpha(2n; n, n, n-1, n-1, \dots, 1, 1) \quad (4.5)$$

algebraically and gave initial thoughts on how a bijective proof could succeed. Let us conclude with a list of related identities – all of them are up to this point conjectured using mathematical computing software. As Theorem 1 provides a combinatorial interpretation of these identities, bijective proofs are of high interest.

**Conjecture 1** ([FR11]). *Let  $n \geq 1$ . Then*

$$\alpha(2n+1; 2n+1, 2n, \dots, 1) = (-1)^n \alpha(n; 2, 4, \dots, 2n) \quad (4.6)$$

*seems to hold, whereby  $\alpha(n; 2, 4, \dots, 2n)$  is known to be the number of Vertically Symmetric ASMs of size  $2n+1$ .*

**Conjecture 2.** *Let  $n \geq 1$ . Then*

$$\alpha(n; 2, 4, \dots, 2n) = \alpha(2n; 2n, 2n, 2n-2, 2n-2, \dots, 2, 2) \quad (4.7)$$

*seems to hold.*

**Conjecture 3.** *Let  $n \geq 1$ . Then*

$$A_n = \alpha(n+i; 1, 2, \dots, i, 1, 2, \dots, n), \quad i = 0, \dots, n, \quad (4.8)$$

$$A_n = (-1)^n \alpha(2n+1; 1, 2, \dots, n+1, 1, 2, \dots, n) \quad (4.9)$$

*seems to hold. Furthermore, the numbers*

$$W_{n,i} = \alpha(2n+1; i, 2, \dots, n+1, 1, 2, \dots, n), \quad i = 1, \dots, 3n+2$$

*seem to satisfy the symmetry  $W_{n,i} = W_{n,3n+3-i}$ .*

**Conjecture 4.** *Let  $n \geq 2$ . Then*

$$A_n = \alpha(n+2; 1, 2, \dots, i+1, i, i+1, \dots, n), \quad i = 1, \dots, n-1 \quad (4.10)$$

*seems to hold.*

Further computational experiments led to the conjecture that (4.5) and (4.10) have the following joint generalization:

**Conjecture 5.** *Let  $n \geq 1$ . Then*

$$A_n = \alpha(n+k; 1, \dots, i-1, i+k-1, i+k-1, i+k-2, i+k-2, \dots, i, i, i+k, i+k+1, \dots, n) \quad (4.11)$$

*seems to hold for  $i = 1, \dots, n-k+1$ ,  $k = 1, \dots, n$ .*

In words, the last identity takes a subsequence  $(i, i+1, \dots, i+k-1)$  of length  $k$  of  $(1, 2, \dots, n)$ , reverses the order, duplicates each entry and puts the subsequence back. Identity (4.5) is thus the



special case of (4.11) where  $k = n$ . Applying (4.1) and the fact that a GMT can not contain three consecutive equal entries, shows that (4.10) is the special case of (4.11) with  $k = 2$ :

$$\begin{aligned} & \alpha(n+2; 1, 2, \dots, i-1, i, i+1, i, i+1, i+2, \dots, n) \\ &= \alpha(n+2; 1, 2, \dots, i-1, i, i, i+1, i+2, \dots, n) + \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i+1, i+2, \dots, n) \\ &= \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i, i+2, \dots, n) + \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i+1, i+1, i+2, \dots, n) \\ &= \alpha(n+2; 1, 2, \dots, i-1, i+1, i+1, i, i, i+2, \dots, n). \end{aligned}$$

From the correspondence between ASMs of size  $n$  and Monotone Triangles with bottom row  $(1, 2, \dots, n)$ , it follows that  $\alpha(n-1; 1, 2, \dots, i-1, i+1, \dots, n)$  is equal to the number of ASMs of size  $n$  with the first row's unique 1 in column  $i$  – denoted  $A_{n,i}$ . In the following conjecture we analogously remove the  $i$ -th argument of the right-hand side in (4.10):

**Conjecture 6.** *Let  $n \geq 1$ . Then*

$$\alpha(n+1; 1, 2, \dots, i-1, i+1, i, i+1, \dots, n) = - \sum_{j=1}^n (j-i) A_{n,j}, \quad i = 1, \dots, n-1 \quad (4.12)$$

*seems to hold.*

As a note on how we found (4.12), let us prove the case  $i = 1$ : Each penultimate row  $(l_1, \dots, l_n)$  of a GMT with bottom row  $(2, 1, 2, \dots, n)$  satisfies  $l_1 = l_2 = 1$  by Condition (3) of GMTs. Taking Conditions (1) and (2) into account, Lemma 1 implies that

$$\alpha(n+1; 2, 1, 2, \dots, n) = - \sum_{p=2}^n \alpha(n; 1, 1, 2, \dots, p-1, p+1, \dots, n).$$

Each penultimate row  $(m_1, \dots, m_{n-1})$  of a GMT with bottom row  $(1, 1, 2, \dots, p-1, p+1, \dots, n)$  satisfies  $m_1 = 1, m_2 = 2, \dots, m_{p-1} = p-1$ . Applying Lemma 1 again yields the claimed equation:

$$\alpha(n+1; 2, 1, 2, \dots, n) = - \sum_{p=2}^n \sum_{j=p}^n A_{n,j} = \sum_{j=2}^n (j-1) A_{n,j}.$$

For general  $i$ , the set of GMTs with bottom row  $(1, 2, \dots, i-1, i+1, i, i+1, \dots, n)$  can be written as disjoint union of those with structure

$$\begin{array}{llllllllllllllll} S_1: & l_1 & & \cdots & & l_{i-2} & & i+1 & & i+1 & & i & & l_{i+2} & & \cdots & & l_n \\ & 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n, \\ S_2: & l_1 & & \cdots & & l_{i-2} & & i+1 & & & & i & & i & & l_{i+2} & & \cdots & & l_n \\ & 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n, \\ S_3: & l_1 & & \cdots & & l_{i-2} & & i-1 & & & & i & & i & & l_{i+2} & & \cdots & & l_n \\ & 1 & & \cdots & & i-2 & & i-1 & & i+1 & & i & & i+1 & & i+2 & & \cdots & & n. \end{array}$$

Similar to the case  $i = 1$ , one can see that the signed enumeration of GMTs with structure  $S_3$  is equal to

$$- \sum_{j=i+1}^n (j-i) A_{n,j}.$$

Proving that the signed enumeration of GMTs with structure  $S_1$  and  $S_2$  yields

$$-\sum_{j=1}^{i-1} (j-i)A_{n,j}$$

remains an open problem. The following conjectures are also related to (4.10) by removing the  $(i-1)$ -st argument of the right-hand side.

**Conjecture 7.** *Let  $n \geq 4$ . Then*

$$\alpha(n+1; 1, 3, 4, 3, 4, 5, \dots, n) = \frac{n+4}{2}A_{n-1} \quad (4.13)$$

*seems to hold.*

As an immediate consequence of Theorem 1 we obtain (the known fact) that the evaluation of  $\alpha(n; k_1, \dots, k_n)$  at integral values is integral. From the definition of ASMs it follows that Vertically Symmetric ASMs only exist for odd size. Therefore, reflection along the vertical symmetry axis is a fixed-point-free involution on the set of even-sized ASMs. So, the number of even-sized ASMs is even and the the right-hand side of (4.13) is an integer too.

Using C. Krattenthaler's Mathematica package RATE, we were able to find more conjectured formulas similar to (4.13):

**Conjecture 8.**

$$\begin{aligned} \alpha(n+1; 1, 2, 4, 5, 4, 5, \dots, n) &= \frac{n^3 + 7n^2 + 10n - 36}{8n - 12}A_{n-1}, \quad n \geq 5, \\ \alpha(n+1; 1, 2, 3, 5, 6, 5, 6, \dots, n) &= \frac{n^4 + 12n^3 + 53n^2 + 54n - 288}{48n - 72}A_{n-1}, \quad n \geq 6. \end{aligned}$$

In general, this leads to the following conjecture:

**Conjecture 9.** *Let  $n \geq k \geq 4$ . Then there exist polynomials  $p_k(n)$  and  $q_k(n)$  with  $\deg p_k - \deg q_k = k - 3$  such that*

$$\alpha(n+1; 1, 2, \dots, k-3, k-1, k, k-1, k, \dots, n) = \frac{p_k(n)}{q_k(n)}A_{n-1}.$$

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